Estimates for parameters and characteristics of the confining SU(3)-gluonic field in an η'-meson

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Abstract.

The confinement mechanism proposed earlier by the author is applied to estimate the possible parameters of the confining SU(3)-gluonic field in an η'-meson. For this aim the electric form factor of an η'-meson is nonperturbatively computed in an explicit analytic form. The estimates obtained are also consistent with the width of the electromagnetic decay $\eta' \to 2\gamma$. The corresponding estimates of the gluon concentrations, electric and magnetic colour field strengths are also adduced for the mentioned field at the scales of the meson under consideration.

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1. Introduction

In Refs. [1, 2, 3] for the Dirac-Yang-Mills system derived from QCD-Lagrangian an unique family of compatible nonperturbative solutions was found and explored, which could pretend to decsribing confinement of two quarks. The applications of the family to the description of both the heavy quarkonia spectra [4, 8] and a number of properties of pions, kaons and η-meson [5, 6, 7] showed that the confinement mechanism is qualitatively the same for both light mesons and heavy quarkonia. At this moment it can be decribed in the following way.

The following main physical reasons underlie linear confinement in the mechanism under discussion. The first one is that gluon exchange between quarks is realized with the propagator different from the photon one, and existence and form of such a propagator is a *direct* consequence of the unique confining nonperturbative solutions of the Yang-Mills equations [2, 3]. The second reason is that, owing to the structure of the mentioned propagator, quarks mainly emit and interchange the soft gluons so the gluon condensate (a classical gluon field) between quarks basically consists of soft gluons (for more details see Refs. [2, 3]) but, because of the fact that any gluon also emits gluons (still softer), the corresponding gluon concentrations rapidly become huge and form a linear confining magnetic colour field of enormous strengths, which leads to confinement of quarks. This is by virtue of the fact that just the magnetic part of the mentioned propagator is responsible for a larger portion of gluon concentrations at large distances since the magnetic part has stronger infrared singularities than the electric one. In the circumstances physically nonlinearity of the Yang-Mills equations effectively vanishes so the latter possess the unique nonperturbative confining solutions of the Abelian-like form (with the values in Cartan subalgebra of SU(3)-Lie algebra) [2, 3] which describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons is soft they cannot leave the hadron (meson) until some gluons obtain additional energy (due to an external reason) to rush out. So we also deal with the confinement of gluons.

The approach under discussion equips us with the explicit wave functions for every two quarks (meson or quarkonium). The wave functions are parametrized by a set of real constants a_j, b_j, B_j describing the mentioned *nonperturbative* confining SU(3)-gluonic field (the gluon condensate) and they are *nonperturbative* modulo square integrable solutions of the Dirac equation in the above confining SU(3)-field and also depend on μ_0, the reduced mass of the current masses of quarks forming meson. It is clear that under the given approach just constants a_j, b_j, B_j, μ_0 determine all properties of any meson (quarkonium), i. e., the approach directly appeals to quark and gluonic degrees of freedom as should be according to the first principles of QCD. Also it is clear that the constants mentioned should be extracted from experimental data.

Such a program has been to a certain extent advanced in Refs. [4, 5, 6, 7, 8]. The aim of the present paper is to continue obtaining estimates for a_j, b_j, B_j for concrete mesons starting from experimental data on spectroscopy of one or another meson. We

here consider an η'-meson and its electromagnetic decay $\eta' \to 2\gamma$.

Of course, when conducting our considerations we shall rely on the standard quark model (SQM) based on SU(3)-flavor symmetry (see, e. g., Ref. [10]), so in accordance with SQM $\eta' = \sqrt{1/3}(\bar{u}u + \bar{d}d + \bar{s}s)$ is a superposition of three quarkonia; consequently, we shall have three sets of parameters a_j, b_j, B_j.

Section 2 contains main relations underlying description of any mesons (quarkonia) in our approach. Section 3 is devoted to computing the electric form factor, the root-mean-square radius $< r >$ and the magnetic moment of the meson under consideration in an explicit analytic form. Section 4 gives an independent estimate for $< r >$ which is used in Section 5 for obtaining estimates for parameters of the confining SU(3)-gluonic field for an η'-meson. Also Section 5 contains a discussion about whether the obtained estimates might also be consistent with the width of 2-photon decay $\eta' \to 2\gamma$. Section 6 employs the obtained parameters of SU(3)-gluonic field to get the corresponding estimates for such characteristics of the mentioned field as gluon concentrations, electric and magnetic colour field strengths at the scales of an η'-meson while Section 7 is devoted to discussion and concluding remarks.

At last, Appendices A and B contain the detailed description of main building blocks for meson wave functions in the approach under discussion, respectively: eigenspinors of the Euclidean Dirac operator on 2-sphere $\mathbb{S}^2$ and radial parts for the modulo square integrable solutions of Dirac equation in the confining SU(3)-Yang-Mills field.

Further we shall deal with the metric of the flat Minkowski spacetime M that we write down (using the ordinary set of local spherical coordinates r, ϑ, φ for the spatial part) in the form

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \equiv dt^2 - dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \,. \quad (1)$$

Besides, we have $|\delta| = |\det(g_{\mu\nu})| = (r^2 \sin\vartheta)^2$ and $0 \le r < \infty$, $0 \le \vartheta < \pi$, $0 \le \varphi < 2\pi$.

Throughout the paper we employ the Heaviside-Lorentz system of units with $\hbar = c = 1$, unless explicitly stated otherwise, so the gauge coupling constant g and the strong coupling constant α_s are connected by the relation $g^2/(4\pi) = \alpha_s$. Further, we shall denote by $L_2(F)$ the set of the modulo square integrable complex functions on any manifold F furnished with an integration measure, then $L_2^n(F)$ will be the n-fold direct product of $L_2(F)$ endowed with the obvious scalar product while $\dagger$ and $*$ stand, respectively, for Hermitian and complex conjugation. Our choice of Dirac γ-matrices conforms to the so-called standard representation and is the same as in Ref. [5]. At last $\otimes$ means tensorial product of matrices and I_n is the unit $n \times n$ matrix so that, e.g., we have

$$I_3 \otimes \gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 & 0 \\ 0 & \gamma^\mu & 0 \\ 0 & 0 & \gamma^\mu \end{pmatrix}$$

for any Dirac γ-matrix γ^μ and so forth.

When calculating we apply the relations 1 GeV^{-1} $\approx$ 0.1973269679 fm , 1 s^{-1} $\approx$ $0.658211915 \times 10^{-24}$ GeV , 1 V/m $\approx 0.2309956375 \times 10^{-23}$ GeV^2, 1 T $= 4\pi \times 10^{-7}$H/m $\times$ 1 A/m $\approx 0.6925075988 \times 10^{-15}$ GeV^2.

Finally, for the necessary estimates we shall employ the T_{00}-component (volumetric energy density) of the energy-momentum tensor for a SU(3)-Yang-Mills field which should be written in the chosen system of units in the form

$$T_{\mu\nu} = -F^a_{\mu\alpha}\, F^a_{\nu\beta}\, g^{\alpha\beta} + \frac{1}{4} F^a_{\beta\gamma}\, F^a_{\alpha\delta} g^{\alpha\beta} g^{\gamma\delta} g_{\mu\nu}\,. \tag{2}$$

2. Main relations

As was mentioned above, our considerations shall be based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system (derived from QCD-Lagrangian) studied at the whole length in Refs. [1, 2, 3]. Referring for more details to those references, let us briefly decribe and specify only the relations necessary to us in the present paper.

One part of the mentioned family is presented by the unique nonperturbative confining solution of the SU(3)-Yang-Mills equations for the gluonic field $A = A_\mu dx^\mu = A^a_\mu \lambda_a dx^\mu$ (λ_a are the known Gell-Mann matrices, $\mu = t, r, \vartheta, \varphi$, $a = 1, ..., 8$) and looks as follows

$$\mathcal{A}_{1t} \equiv A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_1}{r} + A_1\,, \mathcal{A}_{2t} \equiv -A^3_t + \frac{1}{\sqrt{3}} A^8_t = -\frac{a_2}{r} + A_2\,,$$

$$\mathcal{A}_{3t} \equiv -\frac{2}{\sqrt{3}} A^8_t = \frac{a_1 + a_2}{r} - (A_1 + A_2)\,,$$

$$\mathcal{A}_{1\varphi} \equiv A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_1 r + B_1\,, \mathcal{A}_{2\varphi} \equiv -A^3_\varphi + \frac{1}{\sqrt{3}} A^8_\varphi = b_2 r + B_2\,,$$

$$\mathcal{A}_{3\varphi} \equiv -\frac{2}{\sqrt{3}} A^8_\varphi = -(b_1 + b_2) r - (B_1 + B_2) \tag{3}$$

with the real constants a_j, A_j, b_j, B_j parametrizing the family. The word *unique* should be understood in the strict mathematical sense. In fact in Ref. [2] the following theorem was proved:

The unique exact spherically symmetric (nonperturbative) solutions (i.e. depending only on r) of SU(3)-Yang-Mills equations in Minkowski spacetime consist of the family of (3).

It should be noted that solution (3) was found early in Ref. [1] but its uniqueness was proved just in Ref. [2] (see also Ref. [3]). Besides, in Ref. [2] (see also Ref. [5]) it was shown that the above unique confining solutions (3) satisfy the so-called Wilson confinement criterion [9]. Up to now nobody contested this result so if we want to describe interaction between quarks by spherically symmetric SU(3)-fields then they can be only those from the above theorem.

As has been repeatedly explained in Refs. [2, 3, 4, 5], parameters $A_{1,2}$ of solution (3) are inessential for physics in question and we can consider $A_1 = A_2 = 0$. Obviously we have $\sum_{j=1}^3 \mathcal{A}_{jt} = \sum_{j=1}^3 \mathcal{A}_{j\varphi} = 0$ which reflects the fact that for any matrix $\mathcal{T}$ from SU(3)-Lie algebra we have $\mathrm{Tr}\,\mathcal{T} = 0$. Also, as has been repeatedly discussed by us earlier (see, e. g., Refs. [2, 3]), from the above form it is clear that the solution (3) is a configuration describing the electric Coulomb-like colour field (components $A_t^{3,8}$) and the magnetic colour field linear in r (components $A_\varphi^{3,8}$) and we wrote down the solution (3) in the combinations that are just needed further to insert into the Dirac equation (4).

Another part of the family is given by the unique nonperturbative modulo square integrable solutions of the Dirac equation in the confining SU(3)-field of (3) $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ with the four-dimensional Dirac spinors Ψ_j representing the jth colour component of the meson, so Ψ may describe the relative motion (relativistic bound states) of two quarks in mesons and the mentioned Dirac equation is written as follows

$$i\partial_t\Psi \equiv i\begin{pmatrix}\partial_t\Psi_1\\ \partial_t\Psi_2\\ \partial_t\Psi_3\end{pmatrix} = H\Psi \equiv \begin{pmatrix}H_1 & 0 & 0\\ 0 & H_2 & 0\\ 0 & 0 & H_3\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} = \begin{pmatrix}H_1\Psi_1\\ H_2\Psi_2\\ H_3\Psi_3\end{pmatrix}, \tag{4}$$

where the Hamiltonian H_j is

$$H_j = \gamma^0\left[\mu_0 - i\gamma^1\partial_r - i\gamma^2\frac{1}{r}\left(\partial_\vartheta + \frac{1}{2}\gamma^1\gamma^2\right) - i\gamma^3\frac{1}{r\sin\vartheta}\left(\partial_\varphi + \frac{1}{2}\sin\vartheta\gamma^1\gamma^3 + \frac{1}{2}\cos\vartheta\gamma^2\gamma^3\right)\right]$$
$$-g\gamma^0\left(\gamma^0\mathcal{A}_{jt} + \gamma^3\frac{1}{r\sin\vartheta}\mathcal{A}_{j\varphi}\right) \tag{5}$$

with the gauge coupling constant g while μ_0 is a mass parameter and one can consider it to be the reduced mass which is equal, *e. g.*, for quarkonia, to half the current mass of quarks forming a quarkonium.

Then the unique nonperturbative modulo square integrable solutions of (4) are (with Pauli matrix σ_1)

$$\Psi_j = e^{-i\omega_j t}\psi_j \equiv e^{-i\omega_j t}r^{-1}\begin{pmatrix}F_{j1}(r)\Phi_j(\vartheta,\varphi)\\ F_{j2}(r)\sigma_1\Phi_j(\vartheta,\varphi)\end{pmatrix}, j = 1,2,3 \tag{6}$$

with the 2D eigenspinor $\Phi_j = \begin{pmatrix}\Phi_{j1}\\ \Phi_{j2}\end{pmatrix}$ of the Euclidean Dirac operator $\mathcal{D}_0$ on the unit sphere $\mathbb{S}^2$, while the coordinate r stands for the distance between quarks. The explicit form of Φ_j is not needed here and can be found in Refs. [3, 19]. For the purpose of the present paper we shall adduce the necessary spinors in Appendix A. Spinors Φ_j form an orthonormal basis in $L_2^2(\mathbb{S}^2)$. We can call the quantity ω_j the relative energy of the jth colour component of a meson (while ψ_j is the wave function of a stationary state for the jth colour component), but we can see that if we want to interpret (4) as an equation for eigenvalues of the relative motion energy, i. e., to rewrite it in the form $H\psi = \omega\psi$ with $\psi = (\psi_1, \psi_2, \psi_3)$ then we should put $\omega = \omega_j$ for any j so that $H_j\psi_j = \omega_j\psi_j = \omega\psi_j$.

In this situation, if a meson is composed of quarks $q_{1,2}$ with different flavours then the energy spectrum of the meson will be given by $\epsilon = m_{q_1} + m_{q_2} + \omega$ with the current quark masses m_{q_k} (rest energies) of the corresponding quarks. On the other hand for determination of ω_j the following quadratic equation can be obtained [1, 2, 3]

$$[g^2a_j^2+(n_j+\alpha_j)^2]\omega_j^2-2(\lambda_j-gB_j)g^2a_jb_j\,\omega_j+[(\lambda_j-gB_j)^2-(n_j+\alpha_j)^2]g^2b_j^2-\mu_0^2(n_j+\alpha_j)^2=0, \tag{7}$$

which yields

$$\omega_j=\omega_j(n_j,l_j,\lambda_j)=$$
$$\frac{\Lambda_j g^2a_jb_j \pm (n_j+\alpha_j)\sqrt{(n_j^2+2n_j\alpha_j+\Lambda_j^2)\mu_0^2+g^2b_j^2(n_j^2+2n_j\alpha_j)}}{n_j^2+2n_j\alpha_j+\Lambda_j^2}\,, j=1,2,3\,, \tag{8}$$

where $a_3 = -(a_1 + a_2)$, $b_3 = -(b_1 + b_2)$, $B_3 = -(B_1 + B_2)$, $\Lambda_j = \lambda_j - gB_j$, $\alpha_j = \sqrt{\Lambda_j^2 - g^2a_j^2}$, $n_j = 0, 1, 2, ...$, while $\lambda_j = \pm(l_j + 1)$ are the eigenvalues of Euclidean Dirac operator $\mathcal{D}_0$ on a unit sphere with $l_j = 0, 1, 2, ...$ It should be noted that in the papers [1, 2, 3, 4, 5] we used the ansatz (6) with the factor $e^{i\omega_j t}$ instead of $e^{-i\omega_j t}$ but then the Dirac equation (4) would look as $-i\partial_t\Psi = H\Psi$ and in equation (7) the second summand would have the plus sign while the first summand in numerator of (8) would have the minus sign. In the papers [6, 7] we returned to the conventional form of writing Dirac equation and this slightly modified the equations (7)–(8). In the given paper we conform to the same prescription as in Refs. [6, 7].

In line with the above we should have $\omega = \omega_1 = \omega_2 = \omega_3$ in energy spectrum $\epsilon = m_{q_1} + m_{q_2} + \omega$ for any meson (quarkonium) and this at once imposes two conditions on parameters a_j, b_j, B_j when choosing some experimental value for ϵ at the given current quark masses m_{q_1}, m_{q_2}.

The general form of the radial parts of (6) is considered in Appendix B. Within the given paper we need only the radial parts of (6) at $n_j = 0$ (the ground state) that are [see $(B.5)$]

$$F_{j1}=C_jP_jr^{\alpha_j}e^{-\beta_jr}\left(1-\frac{gb_j}{\beta_j}\right), P_j=gb_j+\beta_j,$$
$$F_{j2}=iC_jQ_jr^{\alpha_j}e^{-\beta_jr}\left(1+\frac{gb_j}{\beta_j}\right), Q_j=\mu_0-\omega_j \tag{9}$$

with $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2b_j^2}$, while C_j is determined from the normalization condition $\int_0^\infty(|F_{j1}|^2 + |F_{j2}|^2)dr = \frac{1}{3}$. Consequently, we shall gain that $\Psi_j \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, as a result, the solutions of (6) may describe relativistic bound states (mesons) with the energy (mass) spectrum ϵ.

2.1. Nonrelativistic limit

It is useful to specify the nonrelativistic limit (when $c \to \infty$) for spectrum (8). For this one should replace $g \to g/\sqrt{\hbar c}$, $a_j \to a_j/\sqrt{\hbar c}$, $b_j \to b_j\sqrt{\hbar c}$, $B_j \to B_j/\sqrt{\hbar c}$ and,

expanding (8) in $z = 1/c$, we shall get

$$\omega_j(n_j, l_j, \lambda_j) =$$

$$\pm\mu_0 c^2\left[1 \mp \frac{g^2 a_j^2}{2\hbar^2(n_j+|\lambda_j|)^2} z^2\right] + \left[\frac{\lambda_j g^2 a_j b_j}{\hbar(n_j+|\lambda_j|)^2} \mp \mu_0 \frac{g^3 B_j a_j^2 f(n_j,\lambda_j)}{\hbar^3(n_j+|\lambda_j|)^7}\right] z + O(z^2)\,, \quad (10)$$

where $f(n_j, \lambda_j) = 4\lambda_j n_j(n_j^2+\lambda_j^2) + \frac{|\lambda_j|}{\lambda_j}\left(n_j^4 + 6n_j^2\lambda_j^2 + \lambda_j^4\right)$.

As is seen from (10), at $c \to \infty$ the contribution of linear magnetic colour field (parameters b_j, B_j) to the spectrum really vanishes and the spectrum in essence becomes the purely nonrelativistic Coulomb one (modulo the rest energy). Also it is clear that when $n_j \to \infty$, $\omega_j \to \pm\sqrt{\mu_0^2 + g^2 b_j^2}$. At last, one should specify the weak coupling limit of (8), i.e., the case $g \to 0$. As is not complicated to see from (8), $\omega_j \to \pm\mu_0$ when $g \to 0$. But then quantities $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2} \to 0$ and wave functions of (9) cease to be the modulo square integrable ones at $g = 0$, i.e., they cease to describe relativistic bound states. Accordingly, this means that the equation (8) does not make physical meaning at $g = 0$.

We may seemingly use (8) with various combinations of signes ($\pm$) before the second summand in numerators of (8) but, due to (10), it is reasonable to take all signs equal to plus which is our choice within the paper. Besides, as is not complicated to see, radial parts in the nonrelativistic limit have the behaviour of form $F_{j1}, F_{j2} \sim r^{l_j+1}$, which allows one to call quantum number l_j angular momentum for the jth colour component though angular momentum is not conserved in the field (3) [1, 3]. So, for a meson (quarkonium) under consideration we should put all $l_j = 0$.

2.2. Eigenspinors with $\lambda = \pm 1$

Finally it should be noted that spectrum (8) is degenerated owing to the degeneracy of eigenvalues for the Euclidean Dirac operator $\mathcal{D}_0$ on the unit sphere $\mathbb{S}^2$. Namely, each eigenvlalue of $\mathcal{D}_0$ $\lambda = \pm(l+1), l = 0, 1, 2...$, has multiplicity $2(l+1)$, so we has $2(l+1)$ eigenspinors orthogonal to each other. Ad referendum we need eigenspinors corresponding to $\lambda = \pm 1$ $(l = 0)$ so here is their explicit form [see $(A.16)$]

$$\lambda = -1 : \Phi = \frac{C}{2}\begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2}\begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ -e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2},$$

$$\lambda = 1 : \Phi = \frac{C}{2}\begin{pmatrix} e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2}, \text{ or } \Phi = \frac{C}{2}\begin{pmatrix} -e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2} \quad (11)$$

with the coefficient $C = 1/\sqrt{2\pi}$ (for more details, see Appendix A).

3. Electric form factor, the root-mean-square radius and magnetic moment

Obviously, we should choose a few quantities that are the most important from the physical point of view to characterize meson under consideration and then we

should evaluate the given quantities within the framework of our approach. In the circumstances let us settle on the ground state energy (mass) of an η'-meson, the root-mean-square radius of it and the magnetic moment. All three magnitudes are essentially nonperturbative ones, and can be calculated only by nonperturbative techniques.

Within the present paper we shall use relations (8) at $n_j = 0 = l_j$ so energy (mass) of meson under consideration is given by $\mu = 2m_q + \omega$ with $\omega = \omega_j(0, 0, \lambda_j)$ for any $j = 1, 2, 3$ whereas

$$\omega = \frac{g^2 a_1 b_1}{\Lambda_1} + \frac{\alpha_1 \mu_0}{|\Lambda_1|} = \frac{g^2 a_2 b_2}{\Lambda_2} + \frac{\alpha_2 \mu_0}{|\Lambda_2|} = \frac{g^2 a_3 b_3}{\Lambda_3} + \frac{\alpha_3 \mu_0}{|\Lambda_3|} = \mu - 2m_q \tag{12}$$

and, as a consequence, the corresponding meson (quarkonium) wave functions of (6) are represented by (9) and (11).

3.1. Choice of quark masses and the gauge coupling constant

It is evident for employing the above relations we have to assign some values to quark masses and gauge coupling constant g. In accordance with Ref. [10], at present the current quark masses necessary to us are restricted to intervals $1.5\,\mathrm{MeV} \le m_u \le 3$ MeV, $3.0\,\mathrm{MeV} \le m_d \le 7\,\mathrm{MeV}$, $95\,\mathrm{MeV} \le m_s \le 120\,\mathrm{MeV}$, so we take $m_u = (1.5+3)/2$ MeV $=$ 2.25 MeV, $m_d = (3+7)/2$ MeV $= 5$ MeV, $m_s = (95+120)/2$ MeV $= 107.5$ MeV. Under the circumstances, the reduced mass μ_0 of (5) will respectively take values $m_u/2, m_d/2, m_s/2$. As to gauge coupling constant $g = \sqrt{4\pi\alpha_s}$, it should be noted that recently some attempts have been made to generalize the standard formula for $\alpha_s = \alpha_s(Q^2) = 12\pi/[(33 - 2n_f)\ln(Q^2/\Lambda^2)]$ (n_f is number of quark flavours) holding true at the momentum transfer $\sqrt{Q^2} \to \infty$ to the whole interval $0 \le \sqrt{Q^2} \le \infty$. We shall employ one such a generalization used in Refs. [11]. It is written as follows ($x = \sqrt{Q^2}$ in GeV)

$$\alpha(x) = \frac{12\pi}{(33 - 2n_f)} \frac{f_1(x)}{\ln \frac{x^2 + f_2(x)}{\Lambda^2}} \tag{13}$$

with

$$f_1(x) = 1 + \left(\left(\frac{(1+x)(33-2n_f)}{12} \ln \frac{m^2}{\Lambda^2} - 1 \right)^{-1} + 0.6x^{1.3} \right)^{-1}, \ f_2(x) = m^2(1 + 2.8x^2)^{-2},$$

wherefrom one can conclude that $\alpha_s \to \pi = 3.1415...$ when $x \to 0$, i. e., $g \to 2\pi =$ 6.2831.... We used (13) at $m = 1$ GeV, $\Lambda = 0.234$ GeV, $n_f = 3$, $x = m_{\eta'} = 957.78$ MeV to obtain $g = 3.91476$ necessary for our further computations at the mass scale of an η'-meson.

3.2. Electric form factor

For each meson (quarkonium) with the wave function $\Psi = (\Psi_j)$ of (6) we can define the electromagnetic current $J^\mu = \overline{\Psi}(I_3 \otimes \gamma^\mu)\Psi = (\Psi^\dagger\Psi, \Psi^\dagger(I_3 \otimes \alpha)\Psi) = (\rho, \mathbf{J})$, $\alpha = \gamma^0\gamma$.

The electric form factor $f(K)$ is the Fourier transform of ρ

$$f(K) = \int \Psi^\dagger\Psi e^{-i\mathbf{K}\mathbf{r}} d^3x = \sum_{j=1}^{3} \int \Psi_j^\dagger \Psi_j e^{-i\mathbf{K}\mathbf{r}} d^3x = \sum_{j=1}^{3} f_j(K) =$$

$$\sum_{j=1}^{3} \int (|F_{j1}|^2 + |F_{j2}|^2)\Phi_j^\dagger\Phi_j \frac{e^{-i\mathbf{K}\mathbf{r}}}{r^2} d^3x,\ d^3x = r^2 \sin\vartheta dr d\vartheta d\varphi \tag{14}$$

with the momentum transfer K. At $n_j = 0 = l_j$, as is easily seen, for any spinor of (11) we have $\Phi_j^\dagger\Phi_j = 1/(4\pi)$, so the integrand in (14) does not depend on φ and we can consider vector $\mathbf{K}$ to be directed along z-axis. Then $\mathbf{K}\mathbf{r} = Kr\cos\vartheta$ and with the help of (9) and relations (see Ref. [12]): $\int_0^\infty r^{\alpha-1} e^{-pr} dr = \Gamma(\alpha) p^{-\alpha}$, Re $\alpha, p > 0$, $\int_0^\infty r^{\alpha-1} e^{-pr} \begin{pmatrix} \sin(Kr) \\ \cos(Kr) \end{pmatrix} dr = \Gamma(\alpha)(K^2+p^2)^{-\alpha/2} \begin{pmatrix} \sin(\alpha\arctan(K/p)) \\ \cos(\alpha\arctan(K/p)) \end{pmatrix}$, Re $\alpha > -1$, Re $p > |\mathrm{Im}\, K|$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\int_0^\pi e^{-iKr\cos\vartheta} \sin\vartheta d\vartheta = 2\sin(Kr)/(Kr)$, we shall obtain

$$f(K) = \sum_{j=1}^{3} f_j(K) = \sum_{j=1}^{3} \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \cdot \frac{\sin[2\alpha_j \arctan(K/(2\beta_j))]}{K(K^2+4\beta_j^2)^{\alpha_j}}$$

$$= \sum_{j=1}^{3} \left(\frac{1}{3} - \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2} \cdot \frac{K^2}{6} \right) + O(K^4), \tag{15}$$

wherefrom it is clear that $f(K)$ is a function of K^2, as should be, and we can determine the root-mean-square radius of meson (quarkonium) in the form

$$< r > = \sqrt{\sum_{j=1}^{3} \frac{2\alpha_j^2 + 3\alpha_j + 1}{6\beta_j^2}}. \tag{16}$$

When calculating (15) also the fact was used that by virtue of the normalization condition for wave functions we have $C_j^2[P_j^2(1 - gb_j/\beta_j)^2 + Q_j^2(1 + gb_j/\beta_j)^2] = (2\beta_j)^{2\alpha_j+1}/[3\Gamma(2\alpha_j+1)]$.

It is clear, we can directly calculate $< r >$ in accordance with the standard quantum mechanics rules as $< r > = \sqrt{\int r^2\Psi^\dagger\Psi d^3x} = \sqrt{\sum_{j=1}^{3}\int r^2\Psi_j^\dagger\Psi_j d^3x}$ and the result will be the same as in (16). So we should not call $< r >$ of (16) the *charge* radius of meson (quarkonium)– it is just the radius of meson (quarkonium) determined by the wave functions of (6) (at $n_j = 0 = l_j$) with respect to strong interaction, i.e., radius of confinement. Now we should note the expression (15) to depend on 3-vector $\mathbf{K}$. To rewrite it in the form holding true for any 4-vector Q, let us recall that according to general considerations (see, e.g., Ref. [13]) the relation (15) should correspond to the so-called Breit frame where $Q^2 = -K^2$ [when fixing the metric by (1)] so it is not complicated to rewrite (15) for arbitrary Q in the form

$$f(Q^2) = \sum_{j=1}^{3} f_j(Q^2) = \sum_{j=1}^{3} \frac{(2\beta_j)^{2\alpha_j+1}}{6\alpha_j} \cdot \frac{\sin[2\alpha_j \arctan(\sqrt{|Q^2|}/(2\beta_j))]}{\sqrt{|Q^2|}(4\beta_j^2 - Q^2)^{\alpha_j}} \tag{17}$$

which passes on to (15) in the Breit frame.

3.3. Magnetic moment

We can define the volumetric magnetic moment density by $\mathbf{m} = q(\mathbf{r}\times\mathbf{J})/2 = q[(yJ_z - zJ_y)\mathbf{i} + (zJ_x - xJ_z)\mathbf{j} + (xJ_y - yJ_x)\mathbf{k}]/2$ with the meson charge q and $\mathbf{J} = \Psi^\dagger(I_3\otimes\alpha)\Psi$. Using (6) we have in the explicit form

$$J_x = \sum_{j=1}^{3}(F^*_{j1}F_{j2} + F^*_{j2}F_{j1})\frac{\Phi_j^\dagger\Phi_j}{r^2},\ J_y = \sum_{j=1}^{3}(F^*_{j1}F_{j2} - F^*_{j2}F_{j1})\frac{\Phi_j^\dagger\sigma_2\sigma_1\Phi_j}{r^2},$$

$$J_z = \sum_{j=1}^{3}(F^*_{j1}F_{j2} - F^*_{j2}F_{j1})\frac{\Phi_j^\dagger\sigma_3\sigma_1\Phi_j}{r^2} \tag{18}$$

with Pauli matrices $\sigma_{1,2,3}$. Magnetic moment of meson (quarkonium) is $\mathbf{M} = \int_V \mathbf{m}d^3x$, where V is the volume of the meson (quarkonium) (the ball of radius $< r >$). Then at $n_j = l_j = 0$, as is seen from (9), (11), $F^*_{j1} = F_{j1}, F^*_{j2} = -F_{j2}$, $\Phi_j^\dagger\sigma_2\sigma_1\Phi_j = 0$ for any spinor of (11) which entails $J_x = J_y = 0$, i.e., $m_z = 0$ while $\int_V m_{x,y}d^3x = 0$ because of turning the integral over φ to zero, which is easy to check. As a result, the magnetic moments of mesons (quarkonia) with the wave functions of (6) (at $l_j = 0$) are equal to zero, as should be according to experimental data [10].

Though we can also evaluate the magnetic form factor $F(Q^2)$ of meson (quarkonium) which is also a function of Q^2 (see Refs. [6, 7]) the latter will not be used in the given paper so we shall not dwell upon it.

4. An independent estimate of $< r >$

Inasmuch as at present there exists no generally accepted estimate of $< r >$ for an η'-meson [10], the question now is how to estimate $< r >$ independently and then calculate it within framework of our approach. For this aim, let us employ the present-day width of electromagnetic decay $\eta' \to 2\gamma$ which is approximately equal to $\Gamma_5 = 4.3$ keV according to the notation of Ref. [10]. In this situation, one can use a variant of formulae originating from Ref. [14]. Such formulae are often employed, for example, in the heavy quarkonia physics (see, e. g., Ref. [15]). In their turn, they are actually based on the standard expression from the elementary kinetic theory of gases (see, e. g., Ref. [16]) for the number ν of collisions of a molecule per unit time

$$\nu = \sqrt{2}\sigma < v > n\,, \tag{19}$$

where σ is an effective cross section for molecules, $< v >$ is a mean molecular velocity, n is the concentration of molecules. If replacing $\nu \to \Gamma_5$ we may fit (19) to estimate Γ_5 when interpreting σ as the cross section of annihilation $\bar{q}q \to 2\gamma$ for the quark-antiquark pair due to electromagnetic interaction, $< v >$ and n as, respectively, a mean quark velocity and the concentration of quarks (antiquarks) in meson (quarkonium). To

gain σ in the explicit form one may take the corresponding relation (Dirac formula) for the cross section of annihilation $e^+e^- \to 2\gamma$ (see, e. g., Ref. [13]) and, after replacing $\alpha_{em} \to Q\alpha_{em}$, $m_e \to m_q$ with electromagnetic coupling constant α_{em}=1/137.0359895 and electron mass m_e, we obtain

$$\sigma = \frac{N}{2}\pi r_q^2 \frac{(\tau^2+\tau-\frac{1}{2})\ln\frac{\sqrt{\tau}+\sqrt{\tau-1}}{\sqrt{\tau}-\sqrt{\tau-1}} - (\tau+1)\sqrt{\tau(\tau-1)}}{\tau^2(\tau-1)} \tag{20}$$

with $\tau = s/(4m_q^2)$, where the Mandelstam invariant $s = 2m_q(m_q+\mu/2)$ with $\mu = 957.78$ MeV, $Q = 2/3$ for $\bar{u}u$-state of an η'-meson while $Q = 1/3$ for $\bar{d}d$- and $\bar{s}s$-states, N is the number of colours, $r_q = \alpha_{em}Q^2/m_q$. To get $< v >$ one may use the standard relativistic relation $v = \sqrt{T(T+2E_0)}/(T+E_0)$ with kinetic T and rest energies E_0 for velocity v of a point-like particle. Putting $T = \mu/2 - m_q$, $E_0 = m_q$ we shall gain

$$< v >= \sqrt{1 - \frac{4m_q^2}{\mu^2}}\,. \tag{21}$$

At last, obviously, $n = 1/V$, where the volume of meson (quarkonium) $V = 4\pi < r >^3 /3$ while $< r >$ may be calculated in accordance with (16). The relations (19)–(21) entail the sought independent estimate for $< r >$

$$< r >= \left(\frac{3\sigma\sqrt{2}\sqrt{1-\frac{4m_q^2}{\mu^2}}}{4\pi\Gamma_5}\right)^{1/3} \tag{22}$$

with σ of (20). When inserting $N = 3$, $\mu = 957.78$ MeV, $m_q = 2.25$ MeV, 5 MeV, 107.5 MeV, $\Gamma_5 = 4.3$ keV into (22) we shall have $< r >\approx 19.127$ fm, 4.243 fm, 0.455 fm, respectively, for $\bar{u}u$-, $\bar{d}d$-, $\bar{s}s$-states of an η'-meson. If noticing that for π^0-meson an experimental estimate of $< r >$ is of order 0.672 fm [10], for K^0-meson we have about 0.560 fm [10] while for η-meson $< r >\approx 0.54$ fm [7] then it is reasonable to take $< r >\sim$ (0.40–0.45) fm for η'-meson which is our choice within the present paper. At last, the corresponding quark velocities evaluated in accordance with (21) are $v_u = 0.9999889856$, $v_d = 0.9999456069$, $v_s = 0.9745331899$ so quarks in an η'-meson should be considered the ultrarelativistic point-like particles.

5. Estimates for parameters of SU(3)-gluonic field in η'-meson

5.1. Basic equations and numerical results

Now we are able to estimate parameters a_j, b_j, B_j of the confining SU(3)-field (3) for an η'-meson within framework of our approach. In this situation, we should consider (12) and (16) the system of equations which should be solved compatibly if taking $\mu = 957.78$ MeV, $m_u = 2.25$ MeV, $m_d = 5.0$ MeV, $m_s = 107.5$ MeV and $< r >\approx$ (0.40–0.45) fm (see Section 4). While computing for distinctness we take all the eigenvalues λ_j of the Euclidean Dirac operator $\mathcal{D}_0$ on the unit 2-sphere $\mathbb{S}^2$ equal to 1. The results of numerical compatible solving of equations (12) and (16) are adduced in Tables 1–2.

Table 1. Gauge coupling constant, reduced mass μ_0 and parameters of the confining SU(3)-gluonic field for an η'-meson

Particle	g	μ_0 (MeV)	a_1	a_2	b_1 (GeV)	b_2 (GeV)	B_1	B_2
η'—$\bar{u}u$	3.91476	1.125	0.218474	-0.394718	0.618419	-0.280807	-0.300	-0.200
η'—$\bar{d}d$	3.91476	2.50	0.351384	-0.130858	0.278983	0.285548	-0.150	0.410
η'—$\bar{s}s$	3.91476	53.75	0.123645	0.124633	-0.226875	0.588802	0.410	-0.160

Table 2. Theoretical and experimental η'-meson mass and radius

Particle	Theoret. μ (MeV)	Experim. μ (MeV)	Theoret. $< r >$ (fm)	Experim. $< r >$ (fm)
η'—$\bar{u}u$	$\mu = 2m_u + \omega_j(0,0,1) = 957.78$	957.78	0.404140	-
η'—$\bar{d}d$	$\mu = 2m_d + \omega_j(0,0,1) = 957.78$	957.78	0.404382	-
η'—$\bar{s}s$	$\mu = 2m_s + \omega_j(0,0,1) = 957.78$	957.78	0.362994	-

5.2. Consistency with the width of two-photon decay $\eta' \to 2\gamma$

Let us consider whether the estimates of previous subsection are consistent with the width of the electromagnetic 2-photon decay $\eta' \to 2\gamma$. Actually kinematic analysis based on Lorentz- and gauge invariances gives rise to the following expression for the width Γ of the electromagnetic decay $P \to 2\gamma$ (where P stands for any meson from π^0, η, η', see, e.g., Ref. [17])

$$\Gamma = \frac{1}{4}\pi\alpha_{em}^2 g_{P\gamma\gamma}^2 \mu^3 \tag{23}$$

with the electromagnetic coupling constant α_{em}=1/137.0359895 and the P-meson mass μ while the information about strong interaction of quarks in P-meson is encoded in a decay constant $g_{P\gamma\gamma}$. Making replacement $g_{P\gamma\gamma} = f_P/\mu$ we can reduce (23) to the form

$$\Gamma = \frac{\pi\alpha_{em}^2 \mu f_P^2}{4}\,. \tag{24}$$

Now it should be noted that the only invariant which f_P might depend on is $Q^2 = \mu^2$, i. e. we should find such a function $\mathcal{F}(Q^2)$ for that $\mathcal{F}(Q^2 = \mu^2) = f_P$ but $\mathcal{F}(Q^2)$ cannot be computed by perturbative techniques. It is obvious from the physical point of view that $\mathcal{F}(Q^2)$ should be connected with the electromagnetic properties of P-meson. As we have seen in Section 3, there are at least two suitable functions for this aim – electric and magnetic form factors. But there exist no experimental consequences related to a magnetic form factor at present whereas electric one to some extent determines, e. g., an effective size of meson (quarkonium) in the form $< r >$ of (16). It is reasonable, therefore, to take $\mathcal{F}(Q^2 = \mu^2) = Af(Q^2 = \mu^2)$ with some constant A and the electric form factor f of (17) for the sought relation. In the situation, we obtain an additional equation imposed on parameters of the confining SU(3)-gluonic field in P-meson which

has been used in Refs. [6, 7] to estimate the mentioned parameters in π^0- and η-mesons. As a result, using (17) in the case of an η'-meson, we come from (24) to relation

$$\Gamma = \frac{\pi\alpha_{em}^2\mu}{4}\left(A\sum_{j=1}^{3}\frac{1}{6\alpha_j x_j}\cdot\frac{\sin\left(2\alpha_j\arctan x_j\right)}{(1-x_j^2)^{\alpha_j}}\right)^2 \approx 4.3\,\mathrm{keV} \tag{25}$$

with $x_j = \mu/(2\beta_j)$, $\mu = 957.78$ MeV and we used the width $\Gamma = \Gamma_5 \approx 4.3$ keV for the electromagnetic decay $\eta' \to 2\gamma$ following the notation from Ref. [10]. In the circumstances, we can employ the results of Table 1 and compute the left-hand side of (25) which entails the corresponding value $A \approx 0.231$ for any state of η'—$\bar{u}u$, η'—$\bar{d}d$, η'—$\bar{s}s$. Consequently, we draw the conclusion that parameters of the confining SU(3)-gluonic field in an η'-meson from Table 1 might be consistent with Γ_5.

6. Estimates of gluon concentrations, electric and magnetic colour field strengths

Now let us recall that, according to Refs. [3, 5], one can confront the field (3) with the T_{00}-component (the volumetric energy density of the SU(3)-gluonic field) of the energy-momentum tensor (2) so that

$$T_{00} \equiv T_{tt} = \frac{E^2+H^2}{2} = \frac{1}{2}\left(\frac{a_1^2+a_1a_2+a_2^2}{r^4}+\frac{b_1^2+b_1b_2+b_2^2}{r^2\sin^2\vartheta}\right) \equiv \frac{\mathcal{A}}{r^4}+\frac{\mathcal{B}}{r^2\sin^2\vartheta} \tag{26}$$

with electric E and magnetic H colour field strengths and with real $\mathcal{A} > 0$, $\mathcal{B} > 0$. One can also introduce magnetic colour induction $B = (4\pi\times10^{-7}\mathrm{H/m})\,H$, where H in A/m.

To estimate the gluon concentrations we can employ (26) and, taking the quantity $\omega = \mathit{\Gamma}$, the full decay width of a meson, for the characteristic frequency of gluons we obtain the sought characteristic concentration n in the form

$$n = \frac{T_{00}}{\mathit{\Gamma}}\,, \tag{27}$$

so we can rewrite (26) in the form $T_{00} = T_{00}^{\mathrm{coul}} + T_{00}^{\mathrm{lin}}$ conforming to the contributions from the Coulomb and linear parts of the solution (3). This entails the corresponding split of n from (27) as $n = n_{\mathrm{coul}} + n_{\mathrm{lin}}$.

The parameters of Table 1 were employed when computing and for simplicity we put $\sin\vartheta = 1$ in (26). There was also used the following present-day full decay width of an η'-meson $\mathit{\Gamma} = 0.203$ MeV, whereas the Bohr radius $a_0 = 0.529177249\cdot10^5$ fm [10].

Table 3 contains the numerical results for n_{coul}, n_{lin}, n, E, H, B for the meson under discussion.

7. Discussion and concluding remarks

7.1. Discussion

As is seen from Table 3, at the characteristic scales of an η'-meson the gluon concentrations are huge and the corresponding fields (electric and magnetic colour ones)

Table 3. Gluon concentrations, electric and magnetic colour field strengths in an η'-meson

η'—$\bar{u}u$:	$r_0 =< r >= 0.404140$ fm					
r (fm)	$n_{\rm coul}$ (m^{-3})	$n_{\rm lin}$ (m^{-3})	n (m^{-3})	E (V/m)	H (A/m)	B (T)
$0.1r_0$	0.140954×10^{56}	0.564544×10^{53}	0.141518×10^{56}	0.353468×10^{25}	0.300915×10^{22}	0.378141×10^{16}
r_0	0.140954×10^{52}	0.564544×10^{51}	0.197408×10^{52}	0.353468×10^{23}	0.300915×10^{21}	0.378141×10^{15}
1.0	0.376014×10^{50}	0.922064×10^{50}	0.129808×10^{51}	0.577316×10^{22}	0.121612×10^{21}	0.152822×10^{15}
$10r_0$	0.140954×10^{48}	0.564544×10^{49}	0.578639×10^{49}	0.353468×10^{21}	0.300915×10^{20}	0.378141×10^{14}
a_0	0.479511×10^{31}	0.329275×10^{41}	0.329275×10^{41}	0.206163×10^{13}	0.229813×10^{16}	0.288791×10^{10}
η'—$\bar{d}d$:	$r_0 =< r >= 0.404382$ fm					
r (fm)	$n_{\rm coul}$ (m^{-3})	$n_{\rm lin}$ (m^{-3})	n (m^{-3})	E (V/m)	H (A/m)	B (T)
$0.1r_0$	0.113422×10^{56}	0.468584×10^{53}	0.113891×10^{56}	0.317074×10^{25}	0.274150×10^{22}	0.344507×10^{16}
r_0	0.113422×10^{52}	0.468584×10^{51}	0.160281×10^{52}	0.317074×10^{23}	0.274150×10^{21}	0.344507×10^{15}
1.0	0.303296×10^{50}	0.766251×10^{50}	0.106955×10^{51}	0.518495×10^{22}	0.110861×10^{21}	0.139313×10^{15}
$10r_0$	0.113422×10^{48}	0.468584×10^{49}	0.479926×10^{49}	0.317074×10^{21}	0.274150×10^{20}	0.344507×10^{14}
a_0	0.386778×10^{31}	0.273633×10^{41}	0.273633×10^{41}	0.185158×10^{13}	0.209498×10^{16}	0.263262×10^{10}
η'—$\bar{s}s$:	$r_0 =< r >= 0.362994$ fm					
r (fm)	$n_{\rm coul}$ (m^{-3})	$n_{\rm lin}$ (m^{-3})	n (m^{-3})	E (V/m)	H (A/m)	B (T)
$0.1r_0$	0.853605×10^{55}	0.643673×10^{53}	0.860042×10^{55}	0.275068×10^{25}	0.321312×10^{22}	0.403773×10^{16}
r_0	0.853605×10^{51}	0.643673×10^{51}	0.149728×10^{52}	0.275068×10^{23}	0.321312×10^{21}	0.403773×10^{15}
1.0	0.148202×10^{50}	0.848134×10^{50}	0.996336×10^{50}	0.362443×10^{22}	0.116634×10^{21}	0.146567×10^{15}
$10r_0$	0.853605×10^{47}	0.643673×10^{49}	0.652209×10^{49}	0.275068×10^{21}	0.321312×10^{20}	0.403773×10^{14}
a_0	0.188995×10^{31}	0.302874×10^{41}	0.302874×10^{41}	0.129431×10^{13}	0.220407×10^{16}	0.276972×10^{10}

can be considered to be the classical ones with enormous strenghts. The part $n_{\rm coul}$ of gluon concentration n connected with the Coulomb electric colour field is decreasing faster than $n_{\rm lin}$, the part of n related to the linear magnetic colour field, and at large distances $n_{\rm lin}$ becomes dominant. It should be emphasized that in fact the gluon concentrations are much greater than the estimates given in Table 3 because the latter are the estimates for maximal possible gluon frequencies, i.e. for maximal possible gluon impulses (under the concrete situation of an η'-meson). As was mentioned in Section 1, the overwhelming majority of gluons between quarks is soft, i. e., with frequencies much less than 0.203 MeV so the corresponding concentrations much greater than those in Table 3. The given picture is in concordance with the one obtained in Refs. [4, 5, 6, 7]. As a result, the confinement mechanism developed in Refs. [1, 2, 3] is also confirmed by the considerations of the present paper.

It should be noted, however, that our results are of a preliminary character which is readily apparent, for example, from that the current quark masses (as well as the gauge coupling constant g) used in computation are known only within the certain

limits, and we can expect similar limits for the magnitudes discussed in the paper so it is neccesary for further specification of the parameters for the confining SU(3)-gluonic field in an η'-meson which can be obtained, for instance, by calculating the width of decay $\eta' \to 2\pi^0 + \eta$ with the help of wave functions of η- and π^0-mesons discussed in Refs. [6, 7]. We hope to continue analysing the given problems elsewhere.

7.2. Concluding remarks

Finally we should ask: how to correlate the results obtained above with the relation of SQM mentioned in Section 1

$$\eta' = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s)\,, \tag{28}$$

representing η'-meson as a superposition of three quarkonia. It seems to us one should apply the notions of the pure and mixed states elaborated in quantum mechanics (see, e.g., Ref. [18]). From this point view we may consider an η'-meson the mixed state of three pure states while each pure state is realized with the probability equal to $(1/\sqrt{3})^2 = 1/3$. Then, for example, the root-mean-square radius of an η'-meson should be evaluated according to

$$< r >= \sqrt{\frac{1}{3} < r >^2_{\bar{u}u} + \frac{1}{3} < r >^2_{\bar{d}d} + \frac{1}{3} < r >^2_{\bar{s}s}} \approx 0.390871\,\mathrm{fm} \tag{29}$$

with $< r >_{\bar{q}q}$ adduced in Table 2. Similar remarks obviously hold true for other physical quantities characterizing the meson under consideration.

Appendix A. Eigenspinors of Euclidean Dirac operator on $\mathbb{S}^2$

We here represent some results about eigenspinors of the Euclidean Dirac operator on two-sphere $\mathbb{S}^2$ employed in the main part of the paper.

When separating variables in the Dirac equation (4) there naturally arises the Euclidean Dirac operator $\mathcal{D}_0$ on the unit two-dimensional sphere $\mathbb{S}^2$ and we should know its eigenvalues with the corresponding eigenspinors. Such a problem also arises in the black hole theory while describing the so-called twisted spinors on Schwarzschild and Reissner-Nordström black holes and it was analysed in Refs. [3, 19], so we can use the results obtained therein for our aims. Let us adduce the necessary relations.

The eigenvalue equation for corresponding spinors Φ may look as follows

$$\mathcal{D}_0\Phi = \lambda\Phi. \tag{A.1}$$

As was discussed in Refs. [19], the natural form of $\mathcal{D}_0$ in local coordinates ϑ, φ on the unit sphere $\mathbb{S}^2$ looks as

$$\mathcal{D}_0 = -i\sigma_1\left[i\sigma_2\partial_\vartheta + i\sigma_3\frac{1}{\sin\vartheta}\left(\partial_\varphi - \frac{1}{2}\sigma_2\sigma_3\cos\vartheta\right)\right] =$$

$$\sigma_1\sigma_2\partial_\vartheta+\frac{1}{\sin\vartheta}\sigma_1\sigma_3\partial_\varphi-\frac{\cot\vartheta}{2}\sigma_1\sigma_2 \tag{A.2}$$

with the ordinary Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

so that $\sigma_1\mathcal{D}_0=-\mathcal{D}_0\sigma_1$.

The equation $(A.1)$ was explored in Refs. [19]. Spectrum of D_0 consists of the numbers $\lambda=\pm(l+1)$ with multiplicity $2(l+1)$ of each one, where $l=0,1,2,....$ Let us introduce the number m such that $-l\le m\le l+1$ and the corresponding number $m'=m-1/2$ so $|m'|\le l+1/2$. Then the conforming eigenspinors of operator $\mathcal{D}_0$ are

$$\Phi=\begin{pmatrix}\Phi_1\\\Phi_2\end{pmatrix}=\Phi_{\mp\lambda}=\frac{C}{2}\begin{pmatrix}P^k_{m'-1/2}\pm P^k_{m'1/2}\\P^k_{m'-1/2}\mp P^k_{m'1/2}\end{pmatrix}e^{-im'\varphi} \tag{A.3}$$

with the coefficient $C=\sqrt{\frac{l+1}{2\pi}}$ and $k=l+1/2$. These spinors form an orthonormal basis in $L_2^2(\mathbb{S}^2)$ and are subject to the normalization condition

$$\int_{\mathbb{S}^2}\Phi^\dagger\Phi d\Omega=\int_0^\pi\int_0^{2\pi}(|\Phi_1|^2+|\Phi_2|^2)\sin\vartheta d\vartheta d\varphi=1\,. \tag{A.4}$$

Further, owing to the relation $\sigma_1\mathcal{D}_0=-\mathcal{D}_0\sigma_1$ we, obviously, have

$$\sigma_1\Phi_{\mp\lambda}=\Phi_{\pm\lambda}\,. \tag{A.5}$$

As to functions $P^k_{m'n'}(\cos\vartheta)\equiv P^k_{m',n'}(\cos\vartheta)$ then they can be chosen by miscellaneous ways, for instance, as follows (see, e. g., Ref. [20])

$$P^k_{m'n'}(\cos\vartheta)=i^{-m'-n'}\sqrt{\frac{(k-m')!(k-n')!}{(k+m')!(k+n')!}}\left(\frac{1+\cos\vartheta}{1-\cos\vartheta}\right)^{\frac{m'+n'}{2}}\times$$

$$\times\sum_{j=\max(m',n')}^{k}\frac{(k+j)!i^{2j}}{(k-j)!(j-m')!(j-n')!}\left(\frac{1-\cos\vartheta}{2}\right)^j \tag{A.6}$$

with the orthogonality relation at m',n' fixed

$$\int_0^\pi P^{*k}_{m'n'}(\cos\vartheta)P^{k'}_{m'n'}(\cos\vartheta)\sin\vartheta d\vartheta=\frac{2}{2k+1}\delta_{kk'}\,. \tag{A.7}$$

It should be noted that square of $\mathcal{D}_0$ is

$$\mathcal{D}_0^2=-\Delta_{\mathbb{S}^2}I_2+\sigma_2\sigma_3\frac{\cos\vartheta}{\sin^2\vartheta}\partial_\varphi+\frac{1}{4\sin^2\vartheta}+\frac{1}{4}\,, \tag{A.8}$$

while Laplacian on the unit sphere is

$$\Delta_{\mathbb{S}^2}=\frac{1}{\sin\vartheta}\partial_\vartheta\sin\vartheta\partial_\vartheta+\frac{1}{\sin^2\vartheta}\partial^2_\varphi=\partial^2_\vartheta+\cot\vartheta\partial_\vartheta+\frac{1}{\sin^2\vartheta}\partial^2_\varphi\,, \tag{A.9}$$

so the relation $(A.8)$ is a particular case of the so-called Weitzenböck-Lichnerowicz formulas (see Refs. [21]). Then from $(A.1)$ it follows $\mathcal{D}_0^2\Phi=\lambda^2\Phi$ and, when using the ansatz $\Phi=P(\vartheta)e^{-im'\varphi}=\begin{pmatrix}P_1\\P_2\end{pmatrix}e^{-im'\varphi}$, $P_{1,2}=P_{1,2}(\vartheta)$, the equation $\mathcal{D}_0^2\Phi=\lambda^2\Phi$ turns into

$$\left(-\partial^2_\vartheta-\cot\vartheta\partial_\vartheta+\frac{m'^2+\frac{1}{4}}{\sin^2\vartheta}+\frac{m'\cos\vartheta}{\sin^2\vartheta}\sigma_1\right)P=$$
$$\left(\lambda^2-\frac{1}{4}\right)P\,, \tag{A.10}$$

wherefrom all the above results concerning spectrum of $\mathcal{D}_0$ can be derived [19].

When calculating the functions $P^k_{m'n'}(\cos\vartheta)$ directly, to our mind, it is the most convenient to use the integral expression [20]

$$P^k_{m'n'}(\cos\vartheta)=\frac{1}{2\pi}\sqrt{\frac{(k-m')!(k+m')!}{(k-n')!(k+n')!}}\int_0^{2\pi}\left(e^{i\varphi/2}\cos\frac{\vartheta}{2}+ie^{-i\varphi/2}\sin\frac{\vartheta}{2}\right)^{k-n'}\times$$
$$\left(ie^{i\varphi/2}\sin\frac{\vartheta}{2}+e^{-i\varphi/2}\cos\frac{\vartheta}{2}\right)^{k+n'}e^{im'\varphi}d\varphi \tag{A.11}$$

and the symmetry relations ($z=\cos\vartheta$)

$$P^k_{m'n'}(z)=P^k_{n'm'}(z),\ P^k_{m',-n'}(z)=P^k_{-m',n'}(z),\ P^k_{m'n'}(z)=P^k_{-m',-n'}(z)\,,$$
$$P^k_{m'n'}(-z)=i^{2k-2m'-2n'}P^k_{m',-n'}(z)\,. \tag{A.12}$$

In particular

$$P^k_{kk}(z)=\cos^{2k}(\vartheta/2), P^k_{k,-k}(z)=i^{2k}\sin^{2k}(\vartheta/2), P^k_{k0}(z)=\frac{i^k\sqrt{(2k)!}}{2^kk!}\sin^k\vartheta\,,$$
$$P^k_{kn'}(z)=i^{k-n'}\sqrt{\frac{(2k)!}{(k-n')!(k+n')!}}\sin^{k-n'}(\vartheta/2)\cos^{k+n'}(\vartheta/2)\,. \tag{A.13}$$

Eigenspinors with $\lambda=\pm1,\ \pm2$

If $\lambda=\pm(l+1)=\pm1$ then $l=0$ and from $(A.3)$ it follows that $k=l+1/2=1/2$, $|m'|\le1/2$ and we need the functions $P^{1/2}_{m',\pm1/2}$ that are easily evaluated with the help of $(A.11)$–$(A.13)$ so the eigenspinors for $\lambda=-1$ are

$$\Phi=\frac{C}{2}\begin{pmatrix}\cos\frac{\vartheta}{2}+i\sin\frac{\vartheta}{2}\\ \cos\frac{\vartheta}{2}-i\sin\frac{\vartheta}{2}\end{pmatrix}e^{i\varphi/2},\Phi=\frac{C}{2}\begin{pmatrix}\cos\frac{\vartheta}{2}+i\sin\frac{\vartheta}{2}\\ -\cos\frac{\vartheta}{2}+i\sin\frac{\vartheta}{2}\end{pmatrix}e^{-i\varphi/2}, \tag{A.14}$$

while for $\lambda=1$ the conforming spinors are

$$\Phi=\frac{C}{2}\begin{pmatrix}\cos\frac{\vartheta}{2}-i\sin\frac{\vartheta}{2}\\ \cos\frac{\vartheta}{2}+i\sin\frac{\vartheta}{2}\end{pmatrix}e^{i\varphi/2},\Phi=\frac{C}{2}\begin{pmatrix}-\cos\frac{\vartheta}{2}+i\sin\frac{\vartheta}{2}\\ \cos\frac{\vartheta}{2}+i\sin\frac{\vartheta}{2}\end{pmatrix}e^{-i\varphi/2} \tag{A.15}$$

with the coefficient $C = \sqrt{1/(2\pi)}$.

It is clear that $(A.14)$–$(A.15)$ can be rewritten in the form

$$\lambda = -1 : \Phi = \frac{C}{2}\begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2},\ \text{or}\ \Phi = \frac{C}{2}\begin{pmatrix} e^{i\frac{\vartheta}{2}} \\ -e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2},$$

$$\lambda = 1 : \Phi = \frac{C}{2}\begin{pmatrix} e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2},\ \text{or}\ \Phi = \frac{C}{2}\begin{pmatrix} -e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2}, \qquad (A.16)$$

so the relation $(A.5)$ is easily verified at $\lambda = \pm 1$.

In studying vector mesons and excited states of heavy quarkonia eigenspinors with $\lambda = \pm 2$ may also be useful. Then $k = l + 1/2 = 3/2$, $|m'| \leq 3/2$ and we need the functions $P^{3/2}_{m',\pm 1/2}$ that can be evaluated with the help of $(A.11)$–$(A.13)$. Computation gives rise to

$$P^{3/2}_{3/2,-1/2} = -\frac{\sqrt{3}}{2}\sin\vartheta\sin\frac{\vartheta}{2} = P^{3/2}_{-3/2,1/2},$$

$$P^{3/2}_{3/2,1/2} = i\frac{\sqrt{3}}{2}\sin\vartheta\cos\frac{\vartheta}{2} = P^{3/2}_{-3/2,-1/2},$$

$$P^{3/2}_{1/2,-1/2} = -\frac{i}{4}\left(\sin\frac{\vartheta}{2} - 3\sin\frac{3}{2}\vartheta\right) = P^{3/2}_{-1/2,1/2},$$

$$P^{3/2}_{1/2,1/2} = \frac{1}{4}\left(\cos\frac{\vartheta}{2} + 3\cos\frac{3}{2}\vartheta\right) = P^{3/2}_{-1/2,-1/2}, \qquad (A.17)$$

and according to $(A.3)$ this entails eigenspinors with $\lambda = 2$ in the form

$$\frac{C}{2}i\frac{\sqrt{3}}{2}\sin\vartheta\begin{pmatrix} e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{i3\varphi/2},\ \frac{C}{8}\begin{pmatrix} 3e^{-i\frac{3\vartheta}{2}} + e^{i\frac{\vartheta}{2}} \\ 3e^{i\frac{3\vartheta}{2}} + e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{i\varphi/2},$$

$$\frac{C}{8}\begin{pmatrix} -3e^{-i\frac{3\vartheta}{2}} - e^{i\frac{\vartheta}{2}} \\ 3e^{i\frac{3\vartheta}{2}} + e^{-i\frac{\vartheta}{2}} \end{pmatrix} e^{-i\varphi/2},\ \frac{C}{2}i\frac{\sqrt{3}}{2}\sin\vartheta\begin{pmatrix} -e^{-i\frac{\vartheta}{2}} \\ e^{i\frac{\vartheta}{2}} \end{pmatrix} e^{-i3\varphi/2} \qquad (A.18)$$

with $C = 1/\sqrt{\pi}$, while eigenspinors with $\lambda = -2$ are obtained in accordance with relation $(A.5)$.

Appendix B. Radial parts

We here adduce the explicit form for the radial parts of meson wave functions from (6). At $n_j = 0$ they are given by

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r}\left(1 - \frac{Y_j}{Z_j}\right), F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r}\left(1 + \frac{Y_j}{Z_j}\right), \qquad (B.1)$$

while at $n_j > 0$ by

$$F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r}\left[\left(1 - \frac{Y_j}{Z_j}\right) L^{2\alpha_j}_{n_j}(r_j) + \frac{P_j Q_j}{Z_j} r_j L^{2\alpha_j+1}_{n_j-1}(r_j)\right],$$

$$F_{j2} = iC_jQ_jr^{\alpha_j}e^{-\beta_jr}\left[\left(1+\frac{Y_j}{Z_j}\right)L^{2\alpha_j}_{n_j}(r_j) - \frac{P_jQ_j}{Z_j}r_jL^{2\alpha_j+1}_{n_j-1}(r_j)\right] \qquad (B.2)$$

with the Laguerre polynomials $L^\rho_n(r_j)$, $r_j = 2\beta_jr$, $\beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2b_j^2}$ at $j = 1,2,3$ with $b_3 = -(b_1+b_2)$, $P_j = gb_j + \beta_j$, $Q_j = \mu_0 - \omega_j$, $Y_j = P_jQ_j\alpha_j + (P_j^2 - Q_j^2)ga_j/2$, $Z_j = P_jQ_j\Lambda_j + (P_j^2+Q_j^2)ga_j/2$ with $a_3 = -(a_1+a_2)$, $\Lambda_j = \lambda_j - gB_j$ with $B_3 = -(B_1+B_2)$, $\alpha_j = \sqrt{\Lambda_j^2 - g^2a_j^2}$, while $\lambda_j = \pm(l_j+1)$ are the eigenvalues of Euclidean Dirac operator $\mathcal{D}_0$ on unit two-sphere with $l_j = 0,1,2,...$ (see Appendix A) and quantum numbers $n_j = 0,1,2,...$ are defined by the relations

$$n_j = \frac{gb_jZ_j - \beta_jY_j}{\beta_jP_jQ_j}\,, \qquad (B.3)$$

which entails the quadratic equation (7) and spectrum (8). Further, C_j of $(B.1)$–$(B.2)$ should be determined from the normalization condition

$$\int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2)dr = \frac{1}{3}\,. \qquad (B.4)$$

As a consequence, we shall gain that in (6) $\Psi_j \in L_2^4(\mathbb{R}^3)$ at any $t \in \mathbb{R}$ and, accordingly, $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ may describe relativistic bound states in the field (3) with the energy spectrum (8). As is clear from $(B.3)$ at $n_j = 0$ we have $gb_j/\beta_j = Y_j/Z_j$ so the radial parts of $(B.1)$ can be rewritten as

$$F_{j1} = C_jP_jr^{\alpha_j}e^{-\beta_jr}\left(1 - \frac{gb_j}{\beta_j}\right), F_{j2} = iC_jQ_jr^{\alpha_j}e^{-\beta_jr}\left(1 + \frac{gb_j}{\beta_j}\right)\,. \qquad (B.5)$$

More details can be found in Refs. [1, 3].

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